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ANISOTROPY ELASTIC CONSTITUTIVE RELATION OF THE CRUST MEDIA

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Abstract: Some kinds of anisotropy elastic constitutive relations of the crust media are discussed and a strain energy transformed equation and a calculation formula of the elastic constants of EDA cracks in a non-symmetry plane are also derived. Eigenvalue equations of different anisotropy symmetry systems and their corresponding wave angular dispersion equations are discussed by introducing and deriving equations after previous authors or by the present authors. Finally, some numerical examples are simulated.

Key words: **Anisotropy; Elastic constitutive relation; Angular dispersion equation; P wave; S wave**

1 Introduction

The anisotropy structure of earth media has been researched and applied widely, and various methods and techniques have been established and developed deeply. In geo-science anisotropy study is mainly on seismic waves and conductivity. The seismic waves anisotropy study is developed particularly fast, but there are still some problems in inversions and interpretations. One of which is difficult to correctly recognize the types of anisotropy and to determine uniquely the location of anisotropy media or layers, and the calculated results depend seriously on initial conditions of modeling. Moreover, the most simulations only process transverse isotropy or EDA anisotropy, which seems too simple, compared with practical anisotropy types[1]. The authors think integrating seismic anisotropy and electric anisotropy is a good approach for tackling these problems and it is necessary to consider more complicated anisotropy types. As one part of a series study, this paper mainly discusses anisotropy elastic constitutive relations.

2 Anisotropy types and constitutive relations

2.1 Types of symmetry system and elastic constants

Generally, anisotropy elastic constitutive relation could be written as

$$[\sigma] = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \nu_{yz} \\ \nu_{xz} \\ \nu_{xy} \end{bmatrix} \quad \text{and} \quad c_{ij} = c_{ji} \tag{1}$$

where, σ_i and τ_{ij} are normal stress and shear stress, ε_i and ν_{ij} are normal strain and shear strain, c_{ij} is elastic modulus.

Geometric relation between strain and displacement is

$$\begin{cases} \varepsilon_x = \dfrac{\partial u_x}{\partial x} & \varepsilon_y = \dfrac{\partial u_y}{\partial y} & \varepsilon_z = \dfrac{\partial u_z}{\partial z} \\ \nu_{yz} = \dfrac{\partial u_y}{\partial z} + \dfrac{\partial u_z}{\partial y} & \nu_{xz} = \dfrac{\partial u_z}{\partial_x} + \dfrac{\partial u_x}{\partial z} & \nu_{xy} = \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{cases} \tag{2}$$

where, $u_i\,(i = x, y, z)$ is displacement component. Strain energy equation is

$$W = \frac{1}{2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{yz}\nu_{yz} + \tau_{xz}\nu_{xz} + \tau_{xy}\nu_{xy}) \tag{3}$$

Put equation (1) into (3), the following equation is got:

$$\begin{aligned} W = {} & \frac{1}{2}c_{11}\varepsilon_x^2 + c_{12}\varepsilon_x\varepsilon_y + c_{13}\varepsilon_z\varepsilon_x + c_{14}\nu_{yz}\varepsilon_x + c_{15}\nu_{xz}\varepsilon_x + c_{16}\nu_{xy}\varepsilon_x + \\ & \frac{1}{2}c_{22}\varepsilon_y^2 + c_{23}\varepsilon_z\varepsilon_y + c_{24}\nu_{yz}\varepsilon_y + c_{25}\nu_{xz}\varepsilon_y + c_{26}\nu_{xy}\varepsilon_y + \frac{1}{2}c_{33}\varepsilon_z^2 + c_{34}\nu_{yz}\varepsilon_z + c_{35}\nu_{xz}\varepsilon_z + \\ & c_{36}\nu_{xy}\varepsilon_z + \frac{1}{2}c_{44}\nu_{yz}^2 + c_{45}\nu_{xz}\nu_{yz} + c_{46}\nu_{xy}\nu_{yz} + \frac{1}{2}c_{55}\nu_{xz}^2 + c_{56}\nu_{xy}\nu_{xz} + \frac{1}{2}c_{66}\nu_{xy}^2 \end{aligned} \tag{4}$$

By using equation, it is convenient to discuss elastic modulus matrix through symmetric coordinate systems transformation and rotation, because under any kinds of transformation the strain energy must be kept and for symmetric transformation the structure of elastic modulus matrix must not be changed. Following is an strain energy equation convenient for use. It is noticed that there are some differences between previous for some elastic symmetry system[2,5], the authors are going to check them in following discussions.

Assume coordinate system rotates θ round Z axis, that is

$$\begin{cases} x = x'\cos\theta + y'\sin\theta \\ y = -x'\sin\theta + y'\cos\theta \\ z = z' \end{cases} \quad \text{and} \quad \begin{cases} u_x = u_{x'}\cos\theta + u_{y'}\sin\theta \\ u_y = -u_{x'}\sin\theta + u_{y'}\cos\theta \\ u_z = u_{z'} \end{cases} \tag{5}$$

By using geometric relation between strain and displacement, we have

$$\begin{aligned} W' = {} & \frac{1}{2}c_{11}(\cos^2\theta\varepsilon_x + \sin^2\theta\varepsilon_y - \frac{1}{2}\sin 2\theta\nu_{xy})^2 + c_{12}(\cos^2\theta\varepsilon_x + \sin^2\theta\varepsilon_y - \frac{1}{2}\sin 2\theta\nu_{xy})(\sin^2\theta\varepsilon_x + \cos^2\theta\varepsilon_y + \\ & \frac{1}{2}\sin 2\theta\nu_{xy}) + c_{13}(\cos^2\theta\varepsilon_x + \sin^2\theta\varepsilon_y - \frac{1}{2}\sin 2\theta\nu_{xy})\varepsilon_z + c_{14}(\cos^2\theta\varepsilon_x + \sin^2\theta\varepsilon_y - \frac{1}{2}\sin 2\theta\nu_{xy})(\sin\theta\nu_{xz} + \cos\theta\nu_{yz}) + \\ & c_{15}(\cos^2\theta\varepsilon_x + \sin^2\theta\varepsilon_y - \frac{1}{2}\sin 2\theta\nu_{xy})(\cos\theta\nu_{xz} - \sin\theta\nu_{yz}) + c_{16}(\cos^2\theta\varepsilon_x + \sin^2\theta\varepsilon_y - \frac{1}{2}\sin 2\theta\nu_{xy})[\sin 2\theta(\varepsilon_x - \varepsilon_y) + \\ & \cos 2\theta\nu_{xy}] + \frac{1}{2}c_{22}(\sin^2\theta\varepsilon_x + \cos^2\theta\varepsilon_y + \frac{1}{2}\sin 2\theta\nu_{xy})^2 + c_{23}(\sin^2\theta\varepsilon_x + \cos^2\theta\varepsilon_y + \frac{1}{2}\sin 2\theta\nu_{xy})\varepsilon_z + \\ & c_{24}(\sin^2\theta\varepsilon_x + \cos^2\theta\varepsilon_y + \frac{1}{2}\sin 2\theta\nu_{xy})(\sin\theta\nu_{xz} + \cos\theta\nu_{yz}) + c_{25}(\sin^2\theta\varepsilon_x + \cos^2\theta\varepsilon_y + \frac{1}{2}\sin 2\theta\nu_{xy})(\cos\theta\nu_{xz} - \sin\theta\nu_{yz}) + \\ & c_{26}(\sin^2\theta\varepsilon_x + \cos^2\theta\varepsilon_y + \frac{1}{2}\sin 2\theta\nu_{xy})[\sin 2\theta(\varepsilon_x - \varepsilon_y) + \cos 2\theta\nu_{xy}] + \frac{1}{2}c_{33}\varepsilon_z^2 + c_{34}(\sin\theta\nu_{xz} + \cos\theta\nu_{yz})\varepsilon_z + \\ & c_{35}(\cos\theta\nu_{xz} - \sin\theta\nu_{yz})\varepsilon_z + c_{36}[\sin 2\theta(\varepsilon_x - \varepsilon_y) + \cos 2\theta\nu_{xy}]\varepsilon_z + \frac{1}{2}c_{44}(\sin\theta\nu_{xz} + \cos\theta\nu_{yz})^2 + c_{45}(\sin\theta\nu_{xz} + \cos\theta\nu_{yz}) \cdot \\ & (\cos\theta\nu_{xz} - \sin\theta\nu_{yz}) + c_{46}(\sin\theta\nu_{xz} + \cos\theta\nu_{yz})[\sin 2\theta(\varepsilon_x - \varepsilon_y) + \cos 2\theta\nu_{xy}] + \frac{1}{2}c_{55}(\cos\theta\nu_{xz} - \sin\theta\nu_{yz})^2 + \\ & c_{56}[\sin 2\theta(\varepsilon_x - \varepsilon_y) + \cos 2\theta\nu_{xy}](\cos\theta\nu_{xz} - \sin\theta\nu_{yz}) + \frac{1}{2}c_{66}[\sin 2\theta(\varepsilon_x - \varepsilon_y) + \cos 2\theta\nu_{xy}]^2 \end{aligned} \tag{6}$$

Comparing equation (6) with equation (4) and introduce different rotation angle θ, we get the elastic modulus structures of several main types of anisotropy systems. For example, triangle II symmetry system does not have any symmetry plane but presents symmetry around z axis for each rotation angle 120°, let $\nu_{xy} \neq 0$, and other strain components be zero, and put $\theta = \frac{2}{3}\pi$ and $\theta = \frac{4}{3}\pi$ into equation(6), we have $c_{16} = c_{26} = 0$; let $\varepsilon_x \neq 0$ and other strain components be zero, and let $\varepsilon_y \neq 0$ and other components be zero respectively, we have $c_{11} = c_{12}$ and $c_{66} = \frac{1}{2}(c_{11} - c_{12})$, and so on. Table 1 lists elastic constants matrixes of 9 kinds of symmetric systems (including 2 subsystems) which exists physically and isotropy system. The results are the same as that of Crampin[5], but different from that of Ma et al.[2].

2.2 Common symmetric systems in the crust[2,5]

2.2.1 Isotropy

In homogeneous rocks every plane is symmetric plane and elastic characteristic is the same in every direction. Generally there three cases: medium inherent isotropy without containing cracks or fissures; the cracks are randomly distributed in rocks; crystal lattice or grain are randomly distributed.

2.2.2 Hexagonal symmetry or transverse isotropy

There are two cases. One is transverse isotropy (VTI) with a vertical symmetric axis, which is constructed by periodical thin layers under long wave condition, so such anisotropy is also called PTL, at present the periodical condition has been thought to be unnecessary. In the crust especially in deposition basin, thin layered rocks usually results PTL anisotropy. The other is HTI transverse isotropy or azimuthal anisotropy with a horizontal symmetric axis. Generally such anisotropy is produced by parallel alignment of vertical cracks or fissures. Because in the crust rocks there are extensively distributed saturated cracks in alignment due to stress[6], it is named as Extensive Dilatancy Anisotropy.

2.2.3 Orthorhombic symmetry

It is believed that orthorhombic symmetry exists in orthorhombic crystal olivine, resulting from alignment relative to expending center of the upper mantle. In the deposition basin such anisotropy could be explained as the combination of VTI and HTI or EDA cracks with a horizontal symmetric axis consists of thin layers with a vertical symmetric axis.

2.2.4 Monoclinic symmetry

Such anisotropy consists of two sets of non-vertical crossing aligned cracks, especially on surface of the crust. When observation plane is not vertical to EDA horizontal alignment, the corresponding constants consist of monoclinic symmetry.

3 Wave velocity and angular dispersion

In this section wave velocity and polarization in several common anisotropy media are going to be discussed. First of all Christoffel equation must set up, then let its coefficients determinant be zero, we can get eigenvalue equation of wave propagation[2].

Tab. 1　Anisotropy symmetric systems and elastic constants matrix

Triclinic (no symmetry plane, 21 independent constants) $c_{ij} = c_{ji}$

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{bmatrix}$$

Monoclinic (1 two-fold symmetry axis, 13 independent constants)

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ & c_{22} & c_{23} & 0 & 0 & c_{26} \\ & & c_{33} & 0 & 0 & c_{36} \\ & & & c_{44} & c_{45} & 0 \\ & & & & c_{55} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

Orthorhombic (3 orthogonal two-fold symmetry axes, 9 independent constants)

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{55} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

Trigonal I (1 three-fold symmetry axis and 1 perpendicular two-fold symmetry axis, 6 independent constants)

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & c_{14} \\ & & & & & c_{66} \end{bmatrix}$$

$$2c_{66} = c_{11} - c_{12}$$

Trigonal II (1 three-fold symmetry axis, 7 independent constants)

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & 0 \\ & c_{11} & c_{13} & -c_{14} & -c_{15} & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & -c_{15} \\ & & & & c_{44} & c_{14} \\ & & & & & c_{66} \end{bmatrix}$$

$$2c_{66} = c_{11} - c_{12}$$

Tetragonal I (1 four-fold symmetry axis and 1 perpendicular two-fold symmetry axis, 6 independent constants)

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

Tetragonal II (1 four-fold symmetry axis, 7 independent constants)

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ & c_{11} & c_{13} & 0 & 0 & -c_{16} \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

Cubic (3 perpendicular symmetry planes and their average dividing planes, 3 independent constants)

$$\begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{11} & c_{12} & 0 & 0 & 0 \\ & & c_{11} & 0 & 0 & 0 \\ & & & c_{33} & 0 & 0 \\ & & & & c_{33} & 0 \\ & & & & & c_{33} \end{bmatrix}$$

Hexagonal (1 infinite-fold symmetry axis, 5 independent constants)

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & c_{66} \end{bmatrix}$$

$$2c_{66} = c_{11} - c_{12}$$

Isotropy (2 independent constants)

$$\begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda + 2\mu & 0 & 0 & 0 \\ & & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix}$$

Note 1: Two-fold symmetry axis: symmetry rotation angle is 180°, three-fold symmetry axis: symmetry rotation angle is 120°, four-fold symmetry axis: symmetry rotation angle is 90°, infinite-fold symmetry axis: every rotation angle is symmetry. Here, generally the symmetry axis is z axis.
Note 2: Here only diagonal element and upper trigonal elements are given, the others are symmetry to upper trigonal elements.

The elastic dynamic equation represented by stress (without body force)

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad (i, j = 1, 2, 3) \tag{7}$$

Suppose plane wave with phase velocity v propagates along $\vec{n}$, its displacement represented

$$u_j = p_j \exp[\, i(\vec{x} \cdot \vec{n} - \nu t)] \quad (j = 1, 2, 3) \tag{8}$$

Where p_j is amplitude, ρ is material density and $i^2 = -1$. Put equation (8) into (7), and using constitutive relation and geometry relation between strain and displacement, we get Christoffel equation

$$(\Gamma_{ij} - \rho\nu^2 \delta_{ij}) p_i = 0 \tag{9}$$

where $\Gamma_{ij} = \Gamma_{ji}$, written as

$$\begin{cases} \Gamma_{11} = c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2 + 2c_{56}n_2n_3 + 2c_{15}n_1n_3 + 2c_{16}n_1n_2 \\ \Gamma_{22} = c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2 + 2c_{24}n_2n_3 + 2c_{46}n_1n_3 + 2c_{26}n_1n_2 \\ \Gamma_{33} = c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2 + 2c_{34}n_2n_3 + 2c_{35}n_1n_3 + 2c_{45}n_1n_2 \\ \Gamma_{12} = c_{16}n_1^2 + c_{26}n_2^2 + c_{45}n_3^2 + (c_{25}+c_{46})n_2n_3 + (c_{14}+c_{56})n_1n_3 + (c_{12}+c_{66})n_1n_2 \\ \Gamma_{13} = c_{15}n_1^2 + c_{46}n_2^2 + c_{35}n_3^2 + (c_{36}+c_{45})n_2n_3 + (c_{13}+c_{55})n_1n_3 + (c_{14}+c_{56})n_1n_2 \\ \Gamma_{23} = c_{56}n_1^2 + c_{24}n_2^2 + c_{35}n_3^2 + (c_{23}+c_{44})n_2n_3 + (c_{36}+c_{45})n_1n_3 + (c_{25}+c_{46})n_1n_2 \end{cases} \tag{10}$$

3.1 Isotropy medium

Introduce the constitutive relation of isotropy medium listed in table 1 into equation (9), and let its determinant be zero, we could get eigenvalue equation. From this equation, the solution of velocities are derived:

$$\begin{cases} v_1 = \sqrt{\dfrac{\lambda+2\mu}{\rho}} \\ v_2 = v_3 = \sqrt{\dfrac{\mu}{\rho}} \end{cases} \tag{11}$$

The first velocity is P wave velocity, the second is SV wave velocity, the third is SH wave velocity. It is easy to see that velocities of two shear waves are the same which means splitting phenomenon does not exist.

3.2 PTL anisotropy

Such anisotropy is often called transverse isotropy (TI), because the symmetric axis is vertical, so it is also called VTI. The constitutive relation of such anisotropy has been derived by previous authors[3,4,12,13]. Here we give the constitutive relation directly without derivation:

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \text{ and } 2c_{66} = c_{11} - c_{12} \tag{12}$$

$$c_{11} = h\left(\frac{h_1}{\lambda_1+2\mu_1} + \frac{h_2}{\lambda_2 2\mu_2}\right)^{-1}\left[1 + \frac{4h_1h_2}{h^2}\frac{(\mu_1-\mu_2)(\lambda_1+\mu_1-\lambda_2-\mu_2)}{(\lambda_1+2\mu_1)(\lambda_2+2\mu_2)}\right]$$

$$c_{33} = h\left(\frac{h_1}{\lambda_1+2\mu_1} + \frac{h_2}{\lambda_2+2\mu_2}\right)^{-1},\quad c_{44} = h\left(\frac{h_1}{\mu_1} + \frac{h_2}{\mu_2}\right)^{-1},\quad c_{12} = c_{11} - 2c_{66}$$

$$c_{66} = \frac{h_1\mu_1 + h_2\mu_2}{h},\quad c_{13} = \left(\frac{h_1}{\lambda_1+2\mu_1} + \frac{h_2}{\lambda_2+2\mu_2}\right)^{-1}\left(\frac{h_1\lambda_1}{\lambda_1+2\mu_1} + \frac{h_2\lambda_2}{\lambda_2+2\mu_2}\right)$$

where $h = h_1 + h_2$, $\rho = (h_1\rho_1 + h_2\rho_2)/h$, subscripts 1 and 2 represent two kinds of isotropy media. From equation (9), for such anisotropy medium, we have

$$\begin{vmatrix} c_{11}n_1^2 - \rho v^2 + c_{66}n_2^2 + c_{44}n_3^2 & (c_{11}-c_{66})n_1n_2 & (c_{13}+c_{44})n_1n_3 \\ (c_{11}-c_{66})n_1n_2 & c_{11}n_2^2 - \rho v^2 + c_{66}n_1^2 + c_{44}n_3^2 & (c_{13}+c_{44})n_2n_3 \\ (c_{13}+c_{44})n_1n_3 & (c_{13}+c_{44})n_2n_3 & c_{33}n_3^3 - \rho v^2 + c_{44}n_1^2 + c_{44}n_2^2 \end{vmatrix} = 0 \tag{13}$$

Suppose plane wave propagate in *xoz* plane and $\vec{n} = (\sin\theta, 0, \cos\theta)$, from equation (13), we have

$$\begin{cases} c_{66}\sin^2\theta + c_{44}\cos^2\theta - \rho v^2 = 0 \\ (c_{11}\sin^2\theta + c_{44}\cos^2\theta - \rho v^2)(c_{44}\sin^2\theta + c_{33}\cos^2\theta - \rho v^2) - (c_{13} + c_{44})^2\sin^2\theta\cos^2\theta = 0 \end{cases}$$

Through discussion the velocities of three waves qP, qSV and qSH are got:

$$\begin{cases} \rho V_P^2 = \frac{1}{2}[c_{33} + c_{44} + (c_{11} - c_{33})\sin^2\theta + D(\theta)] \\ \rho V_{SV}^2 = \frac{1}{2}[c_{33} + c_{44} + (c_{11} - c_{33})\sin^2\theta - D(\theta)] \\ \rho V_{SH}^2 = c_{44}\cos^2\theta + c_{66}\sin^2\theta \end{cases} \tag{14}$$

where

$$D(\theta) \equiv \{(c_{33} - c_{44})^2 + 2[2(c_{13} + c_{44})^2 - (c_{33} - c_{44})(c_{11} + c_{33} - 2c_{44})]\sin^2\theta + [(c_{11} + c_{33} - 2c_{44})^2 - 4(c_{13} + c_{44})^2]\sin^4\theta\}^{1/2}$$

It is easy to find that generally two differently polarized S waves have different velocities, their velocity depend on propagation direction, that is angular dispersion or splitting phenomenon. Moreover qP and qSV are coupling except in some special direction. It is also found that along symmetric axis(z axis, $\theta = 0$) the velocities of qSH and qSV are the same without splitting.

3.3 EDA anisotropy

Assume x axis is the normal of EDA cracks, y axis is horizontal strike of EDA cracks and z axis is the vertical alignment of EDA. So x axis is symmetric axis, EDA cracks construct transverse isotropy with a horizontal symmetric axis (HTI or azimuthal anisotropy). Modify the constitutive relation of transverse isotropy listed in table 1, we have EDA constitutive relation:

$$[C] = \begin{bmatrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{12} & c_{23} & c_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \quad \text{and} \quad 2c_{44} = c_{22} - c_{23} \tag{15}$$

According to papers[6, 9~11], EDA elastic constants are

$$[C] = [C^0] + [C^1] + [C^2] \tag{16}$$

Where $[C^0]$ is elastic constants matrix without cracks, the others are 1 and 2 order approximation of cracks density represented respectively as following:

$$[C^1] = -\frac{\varepsilon}{\mu}\begin{bmatrix} (\lambda+2\mu)^2 & \lambda(\lambda+2\mu) & \lambda(\lambda+2\mu) & 0 & 0 & 0 \\ \lambda(\lambda+2\mu) & \lambda^2 & \lambda^2 & 0 & 0 & 0 \\ \lambda(\lambda+2\mu) & \lambda^2 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu^2 \end{bmatrix} D$$

$$[C^2]=\frac{\varepsilon^2}{15(\lambda+2\mu)}\begin{bmatrix}(\lambda+2\mu)^2q & \lambda(\lambda+2\mu)q & \lambda(\lambda+2\mu)q & 0 & 0 & 0\\ \lambda(\lambda+2\mu)q & \lambda^2q & \lambda^2q & 0 & 0 & 0\\ \lambda(\lambda+2\mu)q & \lambda^2q & \lambda^2q & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & X & 0\\ 0 & 0 & 0 & 0 & 0 & X\end{bmatrix}D^2$$

Where
$$\begin{cases} q=15\left(\dfrac{\lambda}{\mu}\right)^2+28\left(\dfrac{\lambda}{\mu}\right)+28\\ X=2\mu(3\lambda+8\mu)\end{cases}$$

D is 6×6 diagonal matrix $[U_{11}, U_{11}, U_{11}, 0, U_{33}, U_{33}]$, and

$$\begin{cases} U_{11}=\dfrac{4}{3}\dfrac{(\lambda+3\mu)}{(\lambda+\mu)}(1+K)^{-1}\\ U_{33}=\dfrac{16}{3}\dfrac{(\lambda+2\mu)}{(3\lambda+4\mu)}(1+M)^{-1}\\ K=\dfrac{\left(k'+\dfrac{4}{3}\mu'\right)}{\pi d\mu}\dfrac{(\lambda+2\mu)}{(\lambda+\mu)}\\ M=\dfrac{4\mu'}{\pi d\mu}\dfrac{(\lambda+2\mu)}{(3\lambda+4\mu)}\\ k'=\lambda'+\dfrac{2}{3}\mu'\end{cases}$$

Here superscript represent elastic parameters of cracked material, non-superscript parameters are solid rock elastic parameters, d indicates aspect ratio ofcrack . Put equation (15)into (9)and let determinant be zero, we have

$$\begin{vmatrix} c_{11}n_1^2-\rho v^2+c_{66}n_2^2+c_{66}n_3^2 & (c_{12}+c_{66})n_1n_2 & (c_{12}+c_{66})n_1n_3\\ (c_{12}+c_{66})n_1n_2 & c_{22}n_2^2-\rho v^2+c_{66}n_1^2+c_{44}n_3^2 & (c_{22}-c_{44})n_2n_3\\ (c_{12}+c_{66})n_1n_3 & (c_{22}-c_{44})n_2n_3 & c_{22}n_3^2-\rho v^2+c_{66}n_1^2+c_{44}n_2^2\end{vmatrix}=0$$

Assume plane waves propagate in xoz plane along $\vec{n}=(\sin\theta, 0, \cos\theta)$, we have

$$\begin{cases} c_{66}\sin^2\theta+c_{44}\cos^2\theta-\rho v^2=0\\ (c_{11}\sin^2\theta+c_{66}\cos^2\theta-\rho v^2)(c_{66}\sin^2\theta+c_{22}\cos^2\theta-\rho v^2)-(c_{12}+c_{66})^2\sin^2\theta\cos^2\theta=0\end{cases}$$

Through discussion the velocities of waves qP, qSV and qSH are got:

$$\begin{cases} \rho V_P^2=\dfrac{1}{2}[c_{22}+c_{66}+(c_{11}-c_{22})\sin^2\theta+D(\theta)]\\ \rho V_{SV}^2=\dfrac{1}{2}[c_{22}+c_{66}+(c_{11}-c_{22})\sin^2\theta-D(\theta)]\\ \rho V_{SH}^2=c_{44}\cos^2\theta+c_{66}\sin^2\theta\end{cases}\tag{17}$$

Where

$$D(\theta)\equiv\{(c_{22}-c_{66})^2+2[2(c_{12}+c_{66})^2-(c_{22}-c_{66})(c_{11}+c_{22}-2c_{66})]\sin^2\theta+[(c_{11}+c_{22}-2c_{66})^2-4(c_{12}+c_{66})^2]\sin^4\theta\}^{1/2}$$

Similar to PTL, of EDA anisotropy generally qSH and qSV have different velocities, but propagating along symmetric axis (x axis, $\theta=90^\circ$) the velocities of qSH and qSV are the same without

splitting.

3.4 Monoclinic system constructed by EDA

When EDA horizontal strike has an angle α with y axis and x axis is not symmetric one, the EDA system constructs a monoclinic system. According to equation (6), 13 elastic constants are derived from equation (15). They are:

$$[C]=\begin{bmatrix} c'_{11} & c'_{12} & c'_{13} & 0 & 0 & c'_{16} \\ c'_{12} & c'_{22} & c'_{23} & 0 & 0 & c'_{26} \\ c'_{13} & c'_{23} & c'_{33} & 0 & 0 & c'_{36} \\ 0 & 0 & 0 & c'_{44} & c'_{45} & 0 \\ 0 & 0 & 0 & c'_{45} & c'_{55} & 0 \\ c'_{16} & c'_{26} & c'_{36} & 0 & 0 & c'_{66} \end{bmatrix} \tag{18}$$

where

$$c'_{11}=\cos^4\alpha\, c_{11}+2\sin^2\alpha\cos^2\alpha\, c_{12}+\sin^4\alpha\, c_{22}+4\sin^2\alpha\cos^2\alpha\, c_{66}$$

$$c'_{12}=\sin^2\alpha\cos^2\alpha\, c_{11}+(\cos^4\alpha+\sin^4\alpha)c_{12}+\sin^2\alpha\cos^2\alpha\, c_{22}-4\sin^2\alpha\cos^2\alpha\, c_{66}$$

$$c'_{13}=\cos^2\alpha\, c_{12}+\sin^2\alpha\, c_{23}$$

$$c'_{16}=\sin\alpha\cos^3\alpha\, c_{11}+(\sin^3\alpha\cos\alpha-\sin\alpha\cos^3\alpha)(c_{12}+2c_{66})-\sin^3\alpha\cos\alpha\, c_{22}$$

$$c'_{22}=\sin^4\alpha\, c_{11}+2\sin^2\alpha\cos^2\alpha\, c_{12}+\cos^4\alpha\, c_{22}+4\sin^2\alpha\cos^2\alpha\, c_{66}$$

$$c'_{23}=\sin^2\alpha\, c_{12}+\cos^2\alpha\, c_{23}$$

$$c'_{26}=\sin^3\alpha\cos\alpha\, c_{11}+(\sin\alpha\cos^3\alpha-\cos\alpha\sin^3\alpha)(c_{12}+2c_{66})-\sin\alpha\cos^3\alpha\, c_{22}$$

$$c'_{33}=c_{33}$$

$$c'_{36}=\sin\alpha\cos\alpha(c_{12}-c_{13})$$

$$c'_{44}=\cos^2\alpha\, c_{44}+\sin^2\alpha\, c_{66}$$

$$c'_{45}=\cos\alpha\sin\alpha(c_{66}-c_{44})$$

$$c'_{55}=\sin^2\alpha\, c_{44}+\cos^2\alpha\, c_{66}$$

$$c'_{66}=\sin^2\alpha\cos^2\alpha\, c_{11}-2\sin^2\alpha\cos^2\alpha\, c_{12}+\sin^2\alpha\cos^2\alpha\, c_{22}+(\cos^2\alpha-\sin^2\alpha)c_{66}$$

3.5 Orthorhombic anisotropy

Such anisotropy consists of EDA thin layers, which is often seen in deposition basin. In our derivation we assume two EDA thin layers construct periodical layers and similarly as Postma (1955)[13], the nine elastic constants for PTL are derived. They are:

$$[C]=\begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \tag{19}$$

Where

$$\begin{cases} c_{11}=\dfrac{h_1c_{11}^1+h_2c_{11}^2}{h}-\dfrac{h_1h_2}{hD}(c_{12}^1-c_{12}^2)^2 & c_{12}=\dfrac{h_1c_{12}^1+h_2c_{12}^2}{h}-\dfrac{h_1h_2}{hD}(c_{12}^1-c_{12}^2)(c_{23}^1-c_{23}^2) \\ c_{13}=\dfrac{h_1c_{12}^1c_{22}^2+h_2c_{12}^2c_{22}^1}{D} & c_{22}=\dfrac{h_1c_{22}^1+h_2c_{22}^2}{h}-\dfrac{h_1h_2}{hD}(c_{23}^1-c_{23}^2)^2 \\ c_{23}=\dfrac{h_1c_{23}^1c_{22}^2+h_2c_{23}^2c_{22}^1}{D} & c_{33}=\dfrac{h}{D}c_{22}^1c_{22}^2 \\ c_{44}=\dfrac{hc_{44}^1c_{44}^2}{h_1c_{44}^2+h_2c_{44}^1} & c_{55}=\dfrac{hc_{66}^1c_{66}^2}{h_1c_{66}^2+h_2c_{66}^1} \\ c_{66}=\dfrac{h_1c_{66}^1+h_2c_{66}^2}{h} & \end{cases}$$

$$\begin{cases} D=h_1c_{22}^2+h_2c_{22}^1 \\ h=h_1+h_2 \\ \rho=\dfrac{h_1\rho_1+h_2\rho_2}{h} \end{cases}$$

In above representation the superscripts show two different EDA layers, h is the sum of depth of two layers. Here we assume the strikes of two EDA layers are the same. Its eigenvalue equation is

$$\begin{vmatrix} c_{11}n_1^2+c_{66}n_2^2+c_{55}n_3^2-\rho v^2 & (c_{12}+c_{66})n_1n_2 & (c_{13}+c_{55})n_1n_3 \\ (c_{12}+c_{66})n_1n_2 & c_{22}n_2^2+c_{66}n_1^2+c_{44}n_3^2-\rho v^2 & (c_{23}+c_{44})n_2n_3 \\ (c_{13}+c_{55})n_1n_3 & (c_{23}+c_{44})n_2n_3 & c_{33}n_3^2+c_{55}n_1^2+c_{44}n_2^2-\rho v^2 \end{vmatrix}=0$$

Assume plane waves propagate in xoz plane along $\vec{n}=(\sin\theta,0,\cos\theta)$, we have

$$\begin{cases} c_{66}\sin^2\theta+c_{44}\cos^2\theta-\rho v^2=0 \\ (c_{11}\sin^2\theta+c_{55}\cos^2\theta-\rho v^2)(c_{55}\sin^2\theta+c_{33}\cos^2\theta-\rho v^2)-(c_{13}+c_{55})^2\sin^2\theta\cos^2\theta=0 \end{cases}$$

The velocity solutions are

$$\begin{cases} \rho V_P^2=\dfrac{1}{2}[c_{33}+c_{55}+(c_{11}-c_{33})\sin^2\theta+D(\theta)] \\ \rho V_{SV}^2=\dfrac{1}{2}[c_{33}+c_{55}+(c_{11}-c_{33})\sin^2\theta-D(\theta)] \\ \rho V_{SH}^2=c_{44}\cos^2\theta+c_{66}\sin^2\theta \end{cases} \tag{20}$$

$$D(\theta)\equiv\{(c_{33}-c_{55})^2+2[2c_{55}(c_{13}+c_{33})+2c_{13}(c_{33}+c_{55})+(c_{11}+c_{33})(c_{55}-c_{33})]\sin^2\theta+[(c_{11}+c_{33})^2-4c_{55}(c_{11}+c_{33})-4c_{13}(c_{13}+2c_{55})]\sin^4\theta\}^{1/2}$$

4 Numerical computation

For description of waves angular dispersion in anisotropy media, here we numerically simulated several angular dispersion cases. The waves propagation plane is xoz plane, propagation direction is $\vec{n}=(\sin\theta,0,\cos\theta)$. Select two elastic material (Table 2) and cracks density to construct several kinds of anisotropy models.

(1) PTL anisotropy: Constructed by two materials listed in Table 2. The relation of three waves and incident angles are shown as Figure 1(a).

Tab. 2　Elastic parameters of numerical computation

Materials	Relative depth	V_P/[km · s^{-1}]	V_S/[km · s^{-1}]	ρ/[g · cm^{-3}]
1	1	4.0	2.2	2.5
2	3	5.8	3.2	2.6

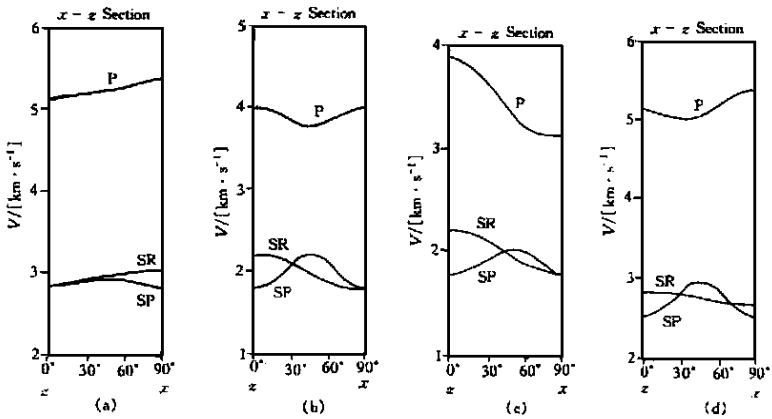

Fig. 1　Angular dispersion of several kinds of anisotropy. Latitude axis indicates wave incident angle related to z axis in x-z plane, 0° represents vertical incidence, 90° represents horizontal incidence.

(2) EDA anisotropy: Constructed by material 1, in first case the cracks are saturated, in the second case the cracks are dried, the cracks density is 0.1. The relations of velocities and incident angles are shown as Figure 1(b) and (c).

(3) EDA-PTL: Constructed by two different EDA thin layers consisting of two materials list in Table 1 with cracks density 0.1. The results are shown as Figure 1(d). If cracks density is zero this case should be the same as the first model.

From the simulated results, it is seen that PTL model, in which V_{SR} is fast than V_{SP}, shows weak splitting than EDA model; of other models in some directions V_{SR} is greater than V_{SP} while in other directions it is reversed, furthermore in some special points two wave velocities are the same without splitting.

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地壳介质的各向异性弹性本构关系讨论

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摘要：论述了地壳介质中存在的几类各向异性弹性本构关系．推导了应变能变换公式、EDA薄层各向异性弹性常数及非对称面上EDA构成的单斜对称弹性常数的计算公式．讨论了几种各向异性对称系统的特征方程，并给出了弹性波的角散表达式．最后选取合理的参数对波的角散现象进行了数值模拟．

关键词：各向异性；弹性本构关系；角散方程；**P**波；**S**波

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